Cubicity of Interval Graphs and the Claw Number

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Abstract. Let $G(V, E)$ be a simple, undirected graph where V is the set of vertices and E is the set of edges. A b-dimensional cube is a Cartesian product $I_1 \times I_2 \times \cdots \times I_b$, where each I_i is a closed interval of unit length on the real line. The *cubicity* of G, denoted by $\mathrm{cub}(G)$ is the minimum positive integer b such that the vertices in G can be mapped to axis parallel b-dimensional cubes in such a way that two vertices are adjacent in G if and only if their assigned cubes intersect. An interval graph is a graph that can be represented as the intersection of intervals on the real line - i.e., the vertices of an interval graph can be mapped to intervals on the real line such that two vertices are adjacent if and only if their corresponding intervals overlap. Suppose $S(m)$ denotes a star graph on $m+1$ nodes. We define *claw number* $\psi(G)$ of the graph to be the largest positive integer m such that $S(m)$ is an induced subgraph of G. It can be easily shown that the cubicity of any graph is at least $\lceil \log_2 \psi(G) \rceil$.
In this paper, we show that, for an interval graph G $\lceil \log_2 \psi(G) \rceil \leq \mathrm{cub}(G) \leq \lceil \log_2 \psi(G) \rceil + 2$. It is not clear whether the upper bound of $\lceil \log_2 \psi(G) \rceil + 2$ is tight: Till now we are unable to find any interval graph with $\mathrm{cub}(G) > \lceil \log_2 \psi(G) \rceil$. We also show that, for an interval graph G, $\mathrm{cub}(G) \leq \lceil \log_2 \alpha \rceil$, where α is the independence number of G. Therefore, in the special case of $\psi(G) = \alpha$, $\mathrm{cub}(G)$ is exactly $\lceil \log_2 \alpha \rceil$.
The concept of cubicity can be generalized by considering boxes instead of cubes. A b-dimensional box is a Cartesian product $I_1 \times I_2 \times \cdots \times I_b$, where each I_i is a closed interval on the real line. The *boxicity* of a graph, denoted $\mathrm{box}(G)$, is the minimum k such that G is the intersection graph of k-dimensional boxes. It is clear that $\mathrm{box}(G) \leq \mathrm{cub}(G)$. From the above result, it follows that for any graph G, $\mathrm{cub}(G) \leq \mathrm{box}(G) \lceil \log_2 \alpha \rceil$.

Keywords: Cubicity, boxicity, interval graphs, indifference graphs, claw number.

1 Introduction

Let $G(V, E)$ be a simple, undirected graph where V is the set of vertices and E is the set of edges. A b-dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_b$, where each R_i is a closed interval on the real line. When each interval has unit length, we will call such a box a b-dimensional cube. The *cubicity* (respectively boxicity) of G, denoted by $\mathrm{cub}(G)$ ($\mathrm{box}(G)$), is the minimum positive integer b such that the vertices in G can be mapped to axis parallel b-dimensional cubes (boxes) in such a way that two vertices are adjacent in G if and only if their assigned cubes (boxes) intersect. Cubicity and boxicity were introduced by Roberts in [20]. Yannakakis [25] proved that it is NP-complete to determine if the cubicity of a graph is at most 3. It was shown by Cozzens [10] that computing the boxicity of a graph is NP-hard. This was strengthened by Kratochvil [18] who showed that deciding whether boxicity of a graph is at most 2 itself is NP-complete.

Roberts [20] showed that for any graph G, $\mathrm{cub}(G) \leq \lfloor 2n/3 \rfloor$ and $\mathrm{box}(G) \leq \lfloor n/2 \rfloor$. The cube representation of special classes of graphs like hypercubes, co-bipartite and complete multipartite graphs were investigated in [7, 25, 20]. Scheinerman [21] showed that the boxicity of outer planar graphs is at most 2. Thomassen [22] proved that the boxicity of planar graphs is at most 3. In [11], Cozzens and Roberts studied the boxicity of split graphs. It is

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interesting to note that coloring problems on low boxicity graphs were considered as early as 1948 [2]. Kostochka [17] provides an extensive survey on colouring problems of intersection graphs. In [16, 13] the complexity of finding the maximum independent set in bounded boxicity graphs is considered. In [15, 12] cubicity has been studied in comparison with sphericity. Some other related references are [1, 4, 6, 8, 9, 23, 24, 19].

In this paper, we consider the cubicity of *interval graphs*. Graphs with boxicity at most 1 are precisely the well-studied class of interval graphs. A graph is an interval graph if and only if its vertices can be mapped to intervals on the real line such that two vertices are adjacent if and only if their corresponding intervals overlap. From the definition of boxicity and cubicity, it is easy to see that any cube representation of a graph will also serve as a box representation. Hence, $\mathrm{box}(G) \leq \mathrm{cub}(G)$. Therefore, it is indeed interesting to ask the following question: what is the cubicity of a graph whose boxicity is 1?

Chandran and Mathew [5] showed that cubicity of an interval graph is at most $\lceil \log_2 |V| \rceil$. This was later improved to $\lceil \log_2 \Delta \rceil + 4$ in [3], where Δ is the maximum degree of G. We improve this bound further. To state our result, we first introduce a parameter called *claw number* of a graph. Recall that a star graph on n vertices is the complete bipartite graph $K_{1,n-1}$. We denote it by $S(n-1)$.

Definition 1. *The* ***claw number*** $\psi(G)$ *of a graph* G *is the largest positive integer* m *such that* $S(m)$ *is an induced subgraph of* G*.*

Our result is as follows:

Theorem 1. *Let* G *be an interval graph with claw number* ψ*.*

$$\lceil \log_2 \psi \rceil \leq \mathrm{cub}(G) \leq \lceil \log_2 \psi \rceil + 2.$$

It is not clear whether the upper bound of $\lceil \log_2 \psi \rceil + 2$ is tight. We have not been able to find any interval graph with cubicity greater than $\lceil \log_2 \psi \rceil$. By slightly modifying the proof of Theorem 1, we can also show that for any interval graph G, $\mathrm{cub}(G) \leq \lceil \log_2 \alpha \rceil$, where α is the independence number of G. Thus, for the special case of $\psi = \alpha$, $\mathrm{cub}(G)$ is exactly $\lceil \log_2 \alpha \rceil$. This in turn allows us to infer that for any graph G, $\mathrm{cub}(G) \leq \mathrm{box}(G) \lceil \log_2 \alpha \rceil$ (See the end of Section 2).

1.1 Some Basic Properties and Results

In this section, we mention some useful properties and results regarding interval graphs and cubicity. A restricted form of interval graphs, that allow only intervals of unit length are called *indifference graphs*. They are also known as *unit interval graphs* or *proper interval graphs*. We provide an alternate definition which we make use of in later sections.

Definition 2. ***Indifference graph:*** *A graph* $G(V, E)$ *is an indifference graph if and only if there exists a function* $\Pi : V \longrightarrow \mathbb{R}$ *such that for two distinct vertices* u *and* v*,* u *and* v *are adjacent if and only if* $|\Pi(u) - \Pi(v)| \leq t$*, for some fixed positive real number* t*.*

It is easy to see that a graph has cubicity 1 if and only if it is an indifference graph.

Property 1. (See Golumbic [14] for a proof.) A graph G is an interval graph if and only if its maximal cliques can be linearly ordered such that for every vertex u the maximal cliques containing u occur consecutively.

For a graph $G(V, E)$, let $G_i(V, E_i)$, $i \in \{1, 2, \ldots, k\}$ be such that $E = E_1 \cap E_2 \cap \cdots \cap E_k$. Then we say that G is the *intersection* of G_i's $1 \leq i \leq k$ and denote it as $G = \bigcap_{i=1}^{k} G_i$. Cubicity (Boxicity respectively) can be stated in terms of intersection of indifference graphs (interval graphs) as follows:

Lemma 1. Roberts [20] *The cubicity (boxicity) of a graph G is the minimum positive integer b such that G is the intersection of b indifference graphs (interval graphs). Moreover, if $G = \bigcap_{i=0}^{m-1} G_i$, for some graphs G_i, then,* $\mathrm{cub}(G) \leq \sum_{i=0}^{m-1} \mathrm{cub}(G_i)$ *and* $\mathrm{box}(G) \leq \sum_{i=0}^{m-1} \mathrm{box}(G_i)$.

The following result is easy to prove.

Lemma 2. *Suppose H is an induced subgraph of G, then* $\mathrm{cub}(G) \geq \mathrm{cub}(H)$.

2 Proof of Theorem 1

The lower bound is easy to see and is as follows. Since the claw number of G is ψ, it has an induced subgraph $S(\psi)$ and $\mathrm{cub}(S(\psi)) = \lceil \log_2 \psi \rceil$ (See Roberts [20]). By Lemma 2, $\mathrm{cub}(G) \geq \mathrm{cub}(S(\psi)) = \lceil \log_2 \psi \rceil$.

Our aim is to construct $\lceil \log_2 \psi \rceil + 2$ indifference graphs and show that G is the intersection of these graphs, thereby proving the upper bound. First, we describe a vertex numbering which is essential for the construction of the indifference graphs.

2.1 Vertex Labelling and the Primary Maximum Independent Set

Let $G(V, E)$ be an interval graph. Let $\mathcal{C} : C_0, C_1, \ldots, C_{k-1}$ correspond to a linear ordering of maximal cliques satisfying Property 1, where C_i corresponds to the set of vertices in the ith maximal clique. For a vertex u, let $c_u = \{i | u \in C_i\}$. It is clear that c_u is a set of consecutive integers. Let $r(u) = \max_{i \in c_u} i$ and $l(u) = \min_{i \in c_u} i$ denote the rightmost and the leftmost cliques containing u respectively. Note that two vertices u and v are adjacent if and only if $c_u \cap c_v \neq \varnothing$.

Let $\eta : V \longrightarrow \mathbb{Z}$ be a labelling of vertices obtained in the following manner: Choose a vertex u_0 such that $r(u_0) \leq r(v)$, $\forall v \neq u_0$. Assign label 0 to u_0 and all vertices adjacent to u_0. Continue the same way considering only the unlabelled vertices until all the vertices are labelled. More formally:

Let $V_0 = V$, $I_{\mathcal{C}} = \varnothing$, $i = 0$;
while $V_i \neq \varnothing$ **do**
  $u_i \in V_i$ be such that $r(u_i) \leq r(v)\ \forall v \in V_i$;
  $V' = \{u_i\} \cup \{v \in V_i | v \text{ is adjacent to } u_i\}$;
  $\eta(w) = i,\ \forall w \in V'$;
  $V_{i+1} = V_i \setminus V'$;
  $I_{\mathcal{C}} \longleftarrow I_{\mathcal{C}} \cup \{u_i\}$;
  $i \longleftarrow i + 1$;
end

Observation 1. *For any vertex v, $\eta(v) \leq i \Longleftrightarrow l(v) \leq r(u_i)$.*

Proof. Since v is adjacent to $u_{\eta(v)}$, we have $l(v) \leq r(u_{\eta(v)})$. It is clear that $r(u_{\eta(v)}) \leq r(u_i)$ since $\eta(v) \leq i$. Therefore, $l(v) \leq r(u_i)$.

Suppose $\eta(v) > i$. From the algorithm, it implies that $r(v) > r(u_i)$. Suppose $l(v) \leq r(u_i)$, that is $l(v) \leq r(u_i) \leq r(v)$. This implies that v is adjacent to u_i. Then, by the algorithm $\eta(v) \leq i$, a contradiction. □

Observation 2. *For two vertices v and w, if $\eta(v) = \eta(w)$, then v and w are adjacent.*

Proof. Let $\eta(v) = \eta(w) = i$. From Observation 1 and from the algorithm it follows that $l(v) \leq r(u_i) \leq r(v)$ and $l(w) \leq r(u_i) \leq r(w)$. Therefore, $r(u_i) \in c_v \cap c_w$. Hence proved.

In the algorithm let l be the number of iterations, i.e. $V_{l-1} \neq \varnothing$ and $V_l = \varnothing$.

Observation 3. *$I_{\mathcal{C}} = \{u_0, u_1, \ldots, u_{l-1}\}$ is a maximum independent set. Hence, $l = \alpha$.*

Proof. From the vertex numbering algorithm it is evident that $\mathcal{C}$ is an independent set. Suppose there exists an independent set of size greater than l. By pigeon hole principle, at least two vertices in this set will be assigned the same number and by Observation 2, they will be adjacent to each other, a contradiction. ◻

$I_{\mathcal{C}}$ is crucial to our construction. From now on we refer to it as the *primary independent set* with respect to the linear ordering $\mathcal{C}$.

Observation 4. $0 = r(u_0) < r(u_1) < \cdots < r(u_{\alpha-1}) = k - 1$.

Proof. From Observation 1 we see that for $i < \alpha - 1$, $r(u_i) < l(u_{i+1}) \leq r(u_{i+1})$. Hence, $r(u_0) < r(u_1) < \cdots < r(u_{\alpha-1})$. Next we show that $r(u_0) = 0$ and $r(u_{\alpha-1}) = k - 1$.

Suppose, $r(u_0) \neq 0$, then it is clear from the algorithm that for all vertices v with $l(v) = 0$, $r(v) > 0$. This implies that C_0 is a subset of C_1, which contradicts the maximality of the cliques.

It is easy to see that $r(u_{\alpha-1}) \leq k - 1$. Suppose $r(u_{\alpha-1}) = t < k - 1$. Consider any vertex $v \in C_{k-1}$. Clearly, $r(v) = k - 1 > t$. Since $\eta(v) \leq \alpha - 1$, from Observation 1, $l(v) \leq t$. Therefore, $l(v) \leq t \leq r(v)$ which implies $v \in C_t$. Hence, $C_{k-1} \subseteq C_t$, which contradicts the maximality of the cliques. ◻

2.2 Defining the Indifference Graphs

Recall that $\mathcal{C} : C_0, C_1, \ldots, C_{k-1}$ is a linear ordering of the maximal cliques of G and $I_{\mathcal{C}} = \{u_0, \ldots, u_{\alpha-1}\}$ is the primary independent set with respect to $\mathcal{C}$. We can assume that $\psi(G) = 2^p$, where p is a positive integer. If not, we will work with another interval graph G' constructed in such a way that $\psi(G') = 2^p$ and G is an induced subgraph of G'. To construct G' from G we consider a vertex $v \in C_{k-1}$. Let m be the largest positive integer such that there exists an induced $S(m)$ in G with v being the central vertex of this $S(m)$. To obtain G', we add $2^p - m$ new vertices $v_0, \ldots, v_{2^p-m-1}$ to G such that they form an independent set and are adjacent to only v. Then it is easy to verify that G' would correspond to the following linear ordering of the maximal cliques: $\mathcal{C}' : C'_0, C'_1, \ldots, C'_{k+2^p-m-1}$, where, $C'_i = C_i$ $0 \leq i \leq k-1$ and $C'_{k+i} = \{v, v_i\}$ $0 \leq i \leq 2^p - m - 1$. Clearly, $\mathcal{C}'$ satisfies Property 1 and therefore G' is an interval graph. Moreover, we have an induced star $S(2^p)$ with v as the central vertex. Clearly, the remaining vertices of G are unaffected by this construction. Hence, $\psi(G') = 2^p$.

Now we define a function $f : \{0, \ldots, k-1\} \longrightarrow \mathbb{R}$ as follows:

1. $f(r(u_0)) = f(0) = 0$.
2. For $j \in \{r(u_i) + 1, \ldots, r(u_{i+1})\}$, $f(j) = i + \frac{1}{2} + \frac{j - r(u_i)}{2(r(u_{i+1}) - r(u_i))}$, for $0 \leq i < \alpha - 1$.

Remark 1. From Observation 4 it is clear that f is defined for each $i \in \{0, 1, \ldots, k-1\}$. Moreover, f is a strictly increasing function.

Given positive integers a and i, the *ith bit function* $b_i(\cdot)$ is defined as $b_i(a) = \left\lfloor \frac{a}{2^i} \right\rfloor \bmod 2$. Now we define another labelling of vertices $\gamma : V \longrightarrow \{0, 1, \ldots, 3\psi - 1\}$ as follows:

$$\gamma(u) = \begin{cases} \eta(u) \bmod \psi + \psi, & \text{if } \left\lfloor \frac{\eta(u)}{\psi} \right\rfloor \text{ is even,} \\ \eta(u) \bmod \psi + 2\psi, & \text{if } \left\lfloor \frac{\eta(u)}{\psi} \right\rfloor \text{ is odd.} \end{cases} \tag{1}$$

Recall that $p = \log_2 \psi$. Note that $\gamma(u)$ is defined in such a way that for $0 \le i \le p-1$, $b_i(\gamma(u)) = b_i(\eta(u))$, i.e. the first p bit positions of $\gamma(u)$ and $\eta(u)$ are identical. The two extra bits in pth and $(p+1)$th positions depend on the parity of $\left\lfloor \frac{\eta(u)}{\psi} \right\rfloor$.

Now, we define $p+2 = \log_2 \psi + 2$ indifference graphs $U_0, U_1, \ldots, U_{p+1}$ as follows. For each U_i we define $\Pi_i : V \longrightarrow \mathbb{R}$ as per Definition 2: For $u \in V$,

$$\Pi_i(u) = \begin{cases} f(r(u)) - \psi + \frac{1}{2}, & \text{if } b_i(\gamma(u)) = 0, \\ f(l(u)), & \text{if } b_i(\gamma(u)) = 1, \end{cases} \tag{2}$$

where $0 \le i \le p+1$. In the graph U_i, two vertices u and v are made adjacent if and only if $|\Pi_i(v) - \Pi_i(u)| \le \psi - \frac{1}{2}$.

2.3 Proof of $G = \bigcap_{i=0}^{p+1} U_i$

Lemma 3. *For any vertex v, $j \in c_v \Longrightarrow f(j) \in \left[\Pi_i(v), \Pi_i(v) + \psi - \frac{1}{2}\right]$, $0 \le i \le p+1$.*

Proof. Let $\eta(v) = m$. In order to handle some boundary cases, we define certain notations. If $q < 0$, then, let $r(u_q) = -1$. If $q > \alpha - 1$, then, let $r(u_q) = r(u_{\alpha-1}) = k - 1$.

Claim 1. $j \in c_v \Longrightarrow r(u_{m-1}) + 1 \le j \le r(u_{m+\psi-1})$.

Proof. If $m = 0$, then it is clear that $l(v) = 0 = r(u_0) = r(u_{-1}) + 1$. Suppose $m > 0$. From Observation 1 it immediately follows that $l(v) \ge r(u_{m-1}) + 1$ and therefore $j > r(u_{m-1})$.

Next, we show that $j \le r(u_{m+\psi-1})$. Suppose $m \ge \alpha - \psi$. Since $q = m + \psi - 1 \ge \alpha - 1$, we have $r(u_{m+\psi-1}) = r(u_q) = r(u_{\alpha-1}) = k - 1$. But trivially, $j \le k - 1$. Hence, we assume that $m < \alpha - \psi$. Suppose $v = u_m$, then this is trivially true from Observation 4. Hence, we assume that $v \ne u_m$. Now, if there exists $j \in c_v$ such that $j > r(u_{m+\psi-1})$, then $t = r(u_{m+\psi-1}) + 1 \in c_v$, since by Observation 1, $l(v) \le r(u_{m+\psi-1})$ and c_v is a set of consecutive integers. There exists a vertex $w \in C_t$ such that $w \notin C_q$, for $q < t$, since otherwise C_t will be a subset of C_{t-1}. Clearly $w \ne v$. Now we claim that $\eta(w) = m + \psi$. Since $l(w) = t > r(u_{m+\psi-1})$, by Observation 1, $\eta(w) \ge m + \psi$. Also $l(u_{m+\psi}) > r(u_{m+\psi-1})$ which implies $r(u_{m+\psi}) \ge l(u_{m+\psi}) \ge t = l(w)$. By the algorithm, $r(w) \ge r(u_{m+\psi})$. Therefore, we have $l(w) \le r(u_{m+\psi}) \le r(w)$ which implies that w is adjacent to $u_{m+\psi}$, which in turn means $\eta(w) = m + \psi$. Since $v, w \in C_t$, they are adjacent. Clearly, the vertex set $V' = \{u_m, u_{m+1}, \ldots, u_{m+\psi-1}, w\}$ forms an independent set since $l(w) = t > r(u_{m+\psi-1})$. Also, all the vertices of V' are adjacent to v since, $l(v) \le r(u_m) \le r(u_{m+\psi-1}) < l(w) \le r(v)$. Therefore, $\{v\} \cup V'$ forms an induced star $S(\psi + 1)$, a contradiction. Hence, $j \le r(u_{m+\psi-1})$. ■

Claim 2. $f(r(v)) - f(l(v)) < \psi - \frac{1}{2}$.

Proof. From the above claim we have $r(u_{m-1}) + 1 \le l(v) \le r(v) \le r(u_{m+\psi-1})$. Now, by the definition of f and noting that f is a strictly increasing function: $\max\left(m - \frac{1}{2}, 0\right) < f(l(v)) \le f(r(v)) \le \min\left(m + \psi - 1, \alpha - 1\right)$. ■

To complete the proof, we need to show that $[f(l(v)), f(r(v))] \subseteq \left[\Pi_i(v), \Pi_i(v) + \psi - \frac{1}{2}\right]$. If $b_i(\gamma(v)) = 0$,

$$\left[\Pi_i(v), \Pi_i(v) + \psi - \frac{1}{2}\right] = \left[f(r(v)) - \psi + \frac{1}{2}, f(r(v))\right],$$

and if $b_i(\gamma(v)) = 1$,

$$\left[\Pi_i(v), \Pi_i(v) + \psi - \frac{1}{2}\right] = \left[f(l(v)), f(l(v)) + \psi - \frac{1}{2}\right].$$

In both cases it is sufficient to show that $f(l(v) > f(r(v)) - \psi + \frac{1}{2}$, which immediately follows from Claim 2. □

Lemma 4. *If $v, w \in V$ such that v and w are adjacent in G, then, v and w are adjacent in all the $p+2$ indifference graphs.*

Proof. Since v and w are adjacent, $c_v \cap c_w \neq \varnothing$. From Lemma 3 it follows that if $j \in c_v \cap c_w$, then, $f(j) \in \left[\mathit{\Pi}_i(v), \mathit{\Pi}_i(v) + \psi - \frac{1}{2}\right] \cap \left[\mathit{\Pi}_i(w), \mathit{\Pi}_i(w) + \psi - \frac{1}{2}\right]$ and hence, $|\mathit{\Pi}_i(v) - \mathit{\Pi}_i(w)| \leq \psi - \frac{1}{2}$ for $0 \leq i \leq p+1$. $\square$

Lemma 5. *If $v, w \in V$ such that v and w are not adjacent in G, then there exists an indifference graph U_i, $i \in \{0, \ldots, p+1\}$, in which u and w are not adjacent.*

Proof. Without loss of generality we assume that $r(v) < l(w)$. Since $l(w) > r(v) \geq r(u_{\eta(v)})$, from Observation 1 it follows that $\eta(v) < \eta(w)$.

Let $q_v = \left\lfloor \frac{\eta(v)}{\psi} \right\rfloor$ and $q_w = \left\lfloor \frac{\eta(w)}{\psi} \right\rfloor$. Now we consider the following cases separately:

1. Suppose $q_w = q_v$: Then, $\gamma(v) \bmod \psi < \gamma(w) \bmod \psi$. This in turn implies that there exists $i < \lceil \log_2 \psi \rceil = p$ such that $b_i(\gamma(v)) = 0$ and $b_i(\gamma(w)) = 1$. Then,
$$\mathit{\Pi}_i(w) - \mathit{\Pi}_i(v) = f(l(w)) - f(r(v)) + \psi - \frac{1}{2} > \psi - \frac{1}{2}.$$
The last inequality follows from the fact that, by definition $f(\cdot)$ is a strictly increasing function.
2. Suppose $q_w = q_v + 1$: If q_v is odd, then $b_p(\gamma(v)) = 0$ and $b_p(\gamma(w)) = 1$ and therefore, as in Case 1, $\mathit{\Pi}_p(w) - \mathit{\Pi}_p(v) > \psi - \frac{1}{2}$. If q_v is even, then $b_{p+1}(\gamma(v)) = 0$ and $b_{p+1}(\gamma(w)) = 1$ and similarly, $\mathit{\Pi}_{p+1}(w) - \mathit{\Pi}_{p+1}(v) > \psi - \frac{1}{2}$.
3. Suppose $q_w = q_v + 2$: If q_v is even, then, $b_p(\gamma(v)) = b_p(\gamma(w)) = 1$. $\mathit{\Pi}_p(w) - \mathit{\Pi}_p(v) = f(l(w)) - f(l(v))$. Note that $\eta(w) \geq q_w \psi$, and therefore, from Observation 1, $l(w) \geq r(u_{q_w\psi - 1}) + 1$. Similarly, $\eta(v) \leq q_v\psi + \psi - 1$, and again from Observation 1, $l(v) \leq r(u_{q_v\psi + \psi - 1})$. Therefore,
$$\begin{aligned} f(l(w)) - f(l(v)) &\geq f(r(u_{q_w\psi-1}) + 1) - f(r(u_{q_v\psi+\psi-1})) \\ &> \left(q_w\psi - 1 + \frac{1}{2}\right) - (q_v\psi + \psi - 1) \\ &= \left(q_v\psi + 2\psi + \frac{1}{2}\right) - (q_v\psi + \psi) \\ &= \psi + \frac{1}{2} > \psi - \frac{1}{2}. \end{aligned}$$
If q_v is odd, then, $b_{p+1}(\gamma(v)) = b_{p+1}(\gamma(w)) = 1$ and in a similar manner as above, we can show that $\mathit{\Pi}_{p+1}(w) - \mathit{\Pi}_{p+1}(v) > \psi - \frac{1}{2}$.
4. Suppose $q_w > q_v + 2$: If $b_p(\gamma(v)) = b_p(\gamma(w)) = 1$, then, we can show that $\mathit{\Pi}_p(w) - \mathit{\Pi}_p(v) > \psi - \frac{1}{2}$ in the same way as Case 3. In a similar way, if $b_{p+1}(\gamma(v)) = b_{p+1}(\gamma(w)) = 1$, we can show that $\mathit{\Pi}_{p+1}(w) - \mathit{\Pi}_{p+1}(v) > \psi - \frac{1}{2}$. Otherwise, from the definition of $\gamma(\cdot)$, it is easy to see that either (1) $b_p(\gamma(v)) = 0$ and $b_p(\gamma(w)) = 1$ OR (2) $b_{p+1}(\gamma(v)) = 0$ and $b_{p+1}(\gamma(w)) = 1$. As in Case 1 we can show that $\mathit{\Pi}_p(w) - \mathit{\Pi}_p(v) > \psi - \frac{1}{2}$ for (1) and $\mathit{\Pi}_{p+1}(w) - \mathit{\Pi}_{p+1}(v) > \psi - \frac{1}{2}$ for (2).

Hence proved. $\square$

Combining Lemmas 4 and 5, we have $G = \bigcap_{i=0}^{p+1} U_i$. Hence, we have proved Theorem 1.

Note that when $\psi = \alpha$, the independence number of G, we have $q_v = 0$ and therefore $b_p(\gamma(v)) = 1$ and $b_{p+1}(\gamma(v)) = 0$ for all vertices $v \in V$. From this, it is easy to see that U_p and U_{p+1} will correspond to complete graphs. Therefore, cubicity of G will be exactly $\lceil \log_2 \alpha \rceil$.

Next we observe that, given any interval graph G, we can construct a graph G' by adding a universal vertex to G. It is easy to that G' is an interval graph which contains G as an induced subgraph. Also, $\psi(G') = \alpha(G') = \alpha$. By Lemma 2, it follows that $\mathrm{cub}(G) \leq \mathrm{cub}(G') = \lceil \log_2 \alpha \rceil$. Considering this, Theorem 1 can be rewritten in the following way:

Theorem 2. *Given an interval graph G, $\lceil \log_2 \psi(G) \rceil \le \mathrm{cub}(G) \le \min(\lceil \log_2 \psi(G) \rceil + 2, \lceil \log_2 \alpha \rceil)$.*

Corollary 1. *Let G be any graph.* $\mathrm{cub}(G) \le \mathrm{box}(G) \lceil \log_2 \alpha \rceil$.

Proof. Let $b = \mathrm{box}(G)$. By Lemma 1, there exist b interval graphs, say G_i, $0 \le i < b$, such that $G = \bigcap_{i=0}^{b-1} G_i$. Since each G_i is a supergraph of G, $\alpha(G_i) \le \alpha$. Therefore, $\mathrm{cub}(G_i) \le \lceil \log_2 \alpha \rceil$. Again by Lemma 1, we have $\mathrm{cub}(G) \le \sum_{i=0}^{b-1} \mathrm{cub}(G_i) \le \mathrm{box}(G) \lceil \log_2 \alpha \rceil$. ◻

We observe that this inequality is tight. In fact, given any two positive integers k and l, there exists a graph G with $\mathrm{box}(G) = k$, $\alpha = l$ such that $\mathrm{cub}(G) = k \lceil \log_2 l \rceil$. One such example is the complete k-partite graph with $|V| = kl$ (See Roberts [20] for proofs).

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